Introduction to
Global Navigation Satellite System (GNSS)
Module: 2

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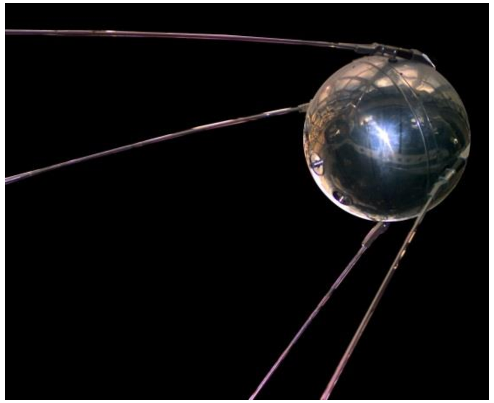
Module 2: Course Contents

- Satellite Orbits
- Navigation Data Format
- Position Computation
- Output Data Formats

Orbital Mechanics

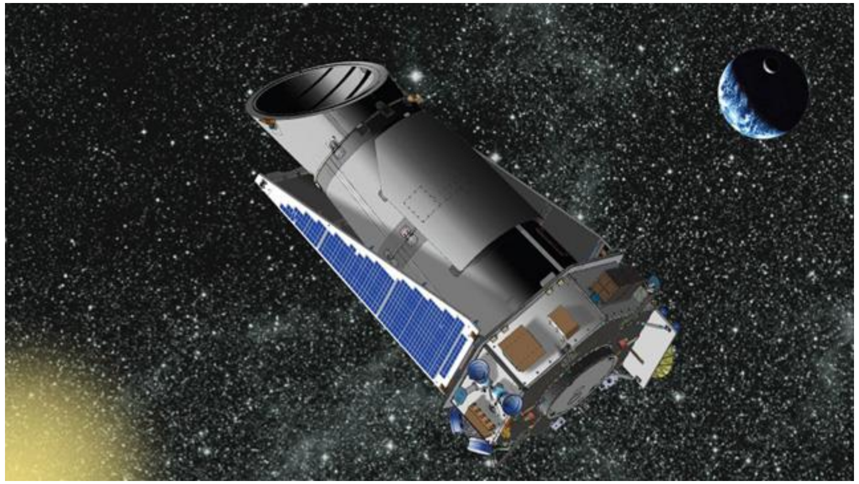
- **Orbital mechanics** or astrodynamics is the application of celestial mechanics to the practical problem concerning the motion of spacecraft.
  - A core discipline within space mission design, control, and operation.
- **Celestial mechanics** treats the orbital dynamics of natural astronomical bodies such as star systems, planets, and moons.

Sputnik-1
The first artificial Earth satellite launched by the Soviet Union in 1957.



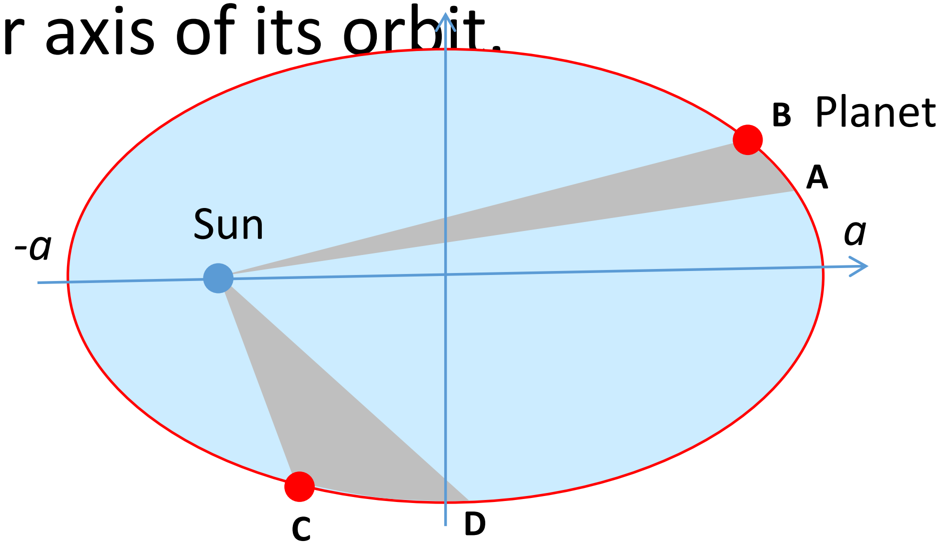
History

- There is little distinction between orbital and celestial mechanics. The fundamental techniques are the same.
- **Johannes Kepler** was the first to successfully model planetary orbits to a high degree of accuracy, publishing his laws of planetary motion in 1605.



Kepler's Laws of Planet Motion

- The orbit of every planet is an **ellipse** with the Sun at one of the two foci (plural of focus).
- A line joining a planet and the sun sweeps out equal area during equal intervals of time.
- The square of the orbital period of planet is proportional to the cube of the semi-major axis of its orbit.



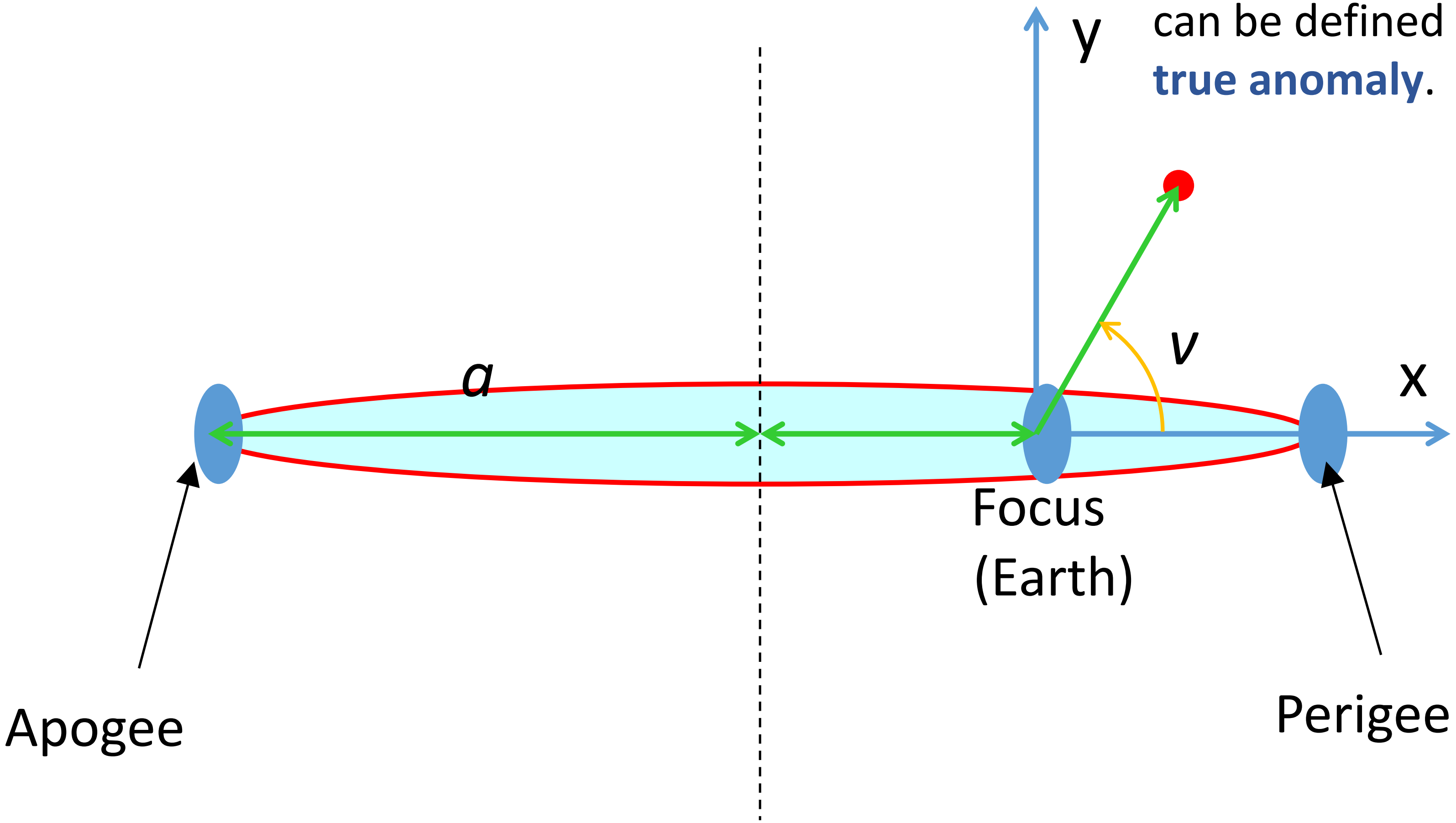
Kepler Orbit

- Kepler orbit can be uniquely defined by six parameters know as **Keplerian elements**.
  - **Semi-major axis** (*a*)
  - **Eccentricity** (*e*)
  - **Inclination** (*i*)
  - **Right ascension of the ascending node** (RAAN) (Ω)
  - **Argument of perigee** (ω)
  - **True anomaly** (v: Greek letters *nu*)

Orbital Plane

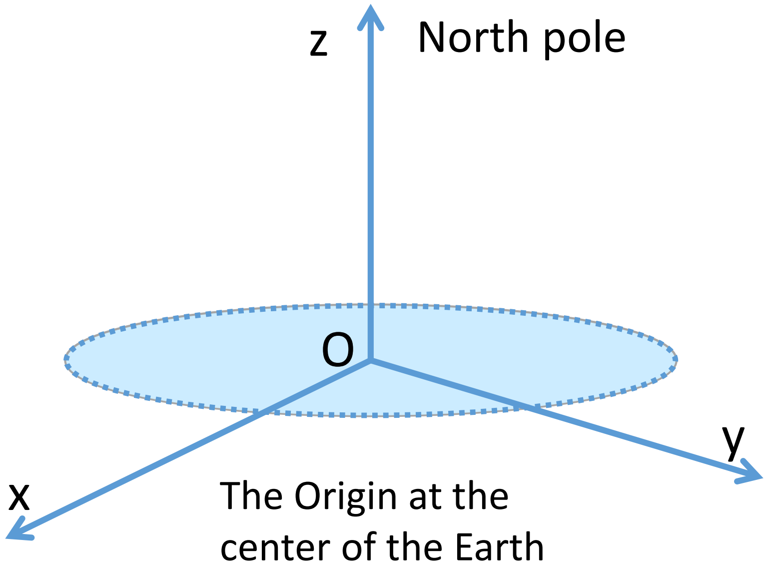
The shape of an elliptic orbit can be defined by the **semi-major axis** and **eccentricity**.

The satellite position in the orbital plane can be defined by **true anomaly**.



Equatorial Coordinate System

- The most common coordinate frame for describing satellite orbits is the geocentric equatorial coordinate system, which is also called an Earth-Centered Inertial (**ECI**) coordinate system.



Vernal equinox

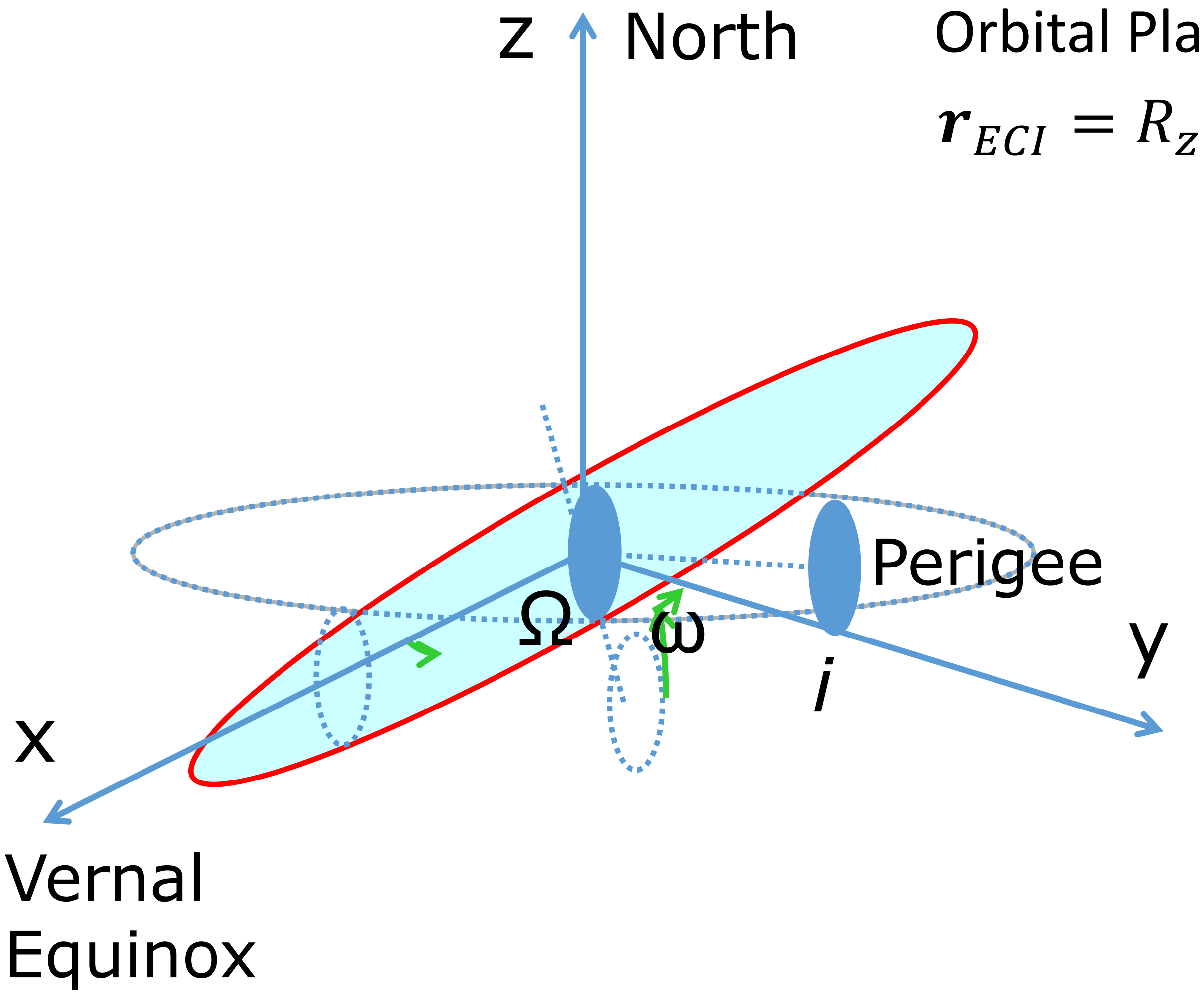
The direction of the Sun as seen from Earth at the beginning of spring time.

Equatorial plane

The fundamental plane is consisting of the projection of the Earth's equator

The Origin at the center of the Earth

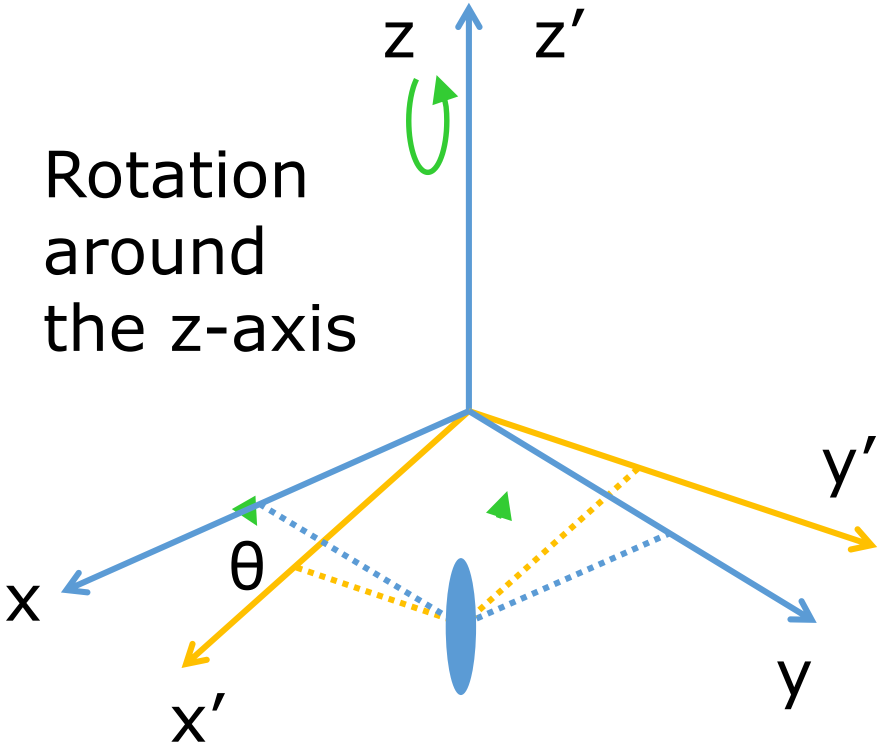
Orientation of the Orbital Plane



Orbital Plane to ECI

$$\boldsymbol{r}_{ECI} = R_z(-\Omega)R_x(-i)R_z(-\omega)\boldsymbol{r}$$

Rotation Matrices



Rotation around the z-axis

$$R_z(\theta)=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

Around the x-axis

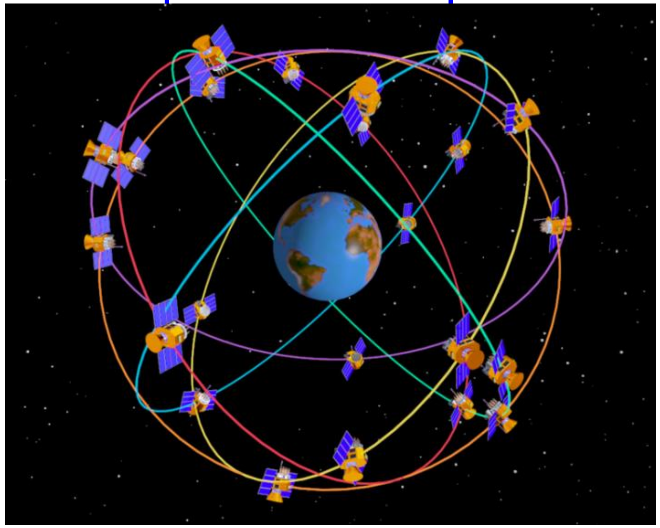
$$R_x(\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}$$

Around the y-axis

$$R_y(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$$

Typical GPS Orbit

- 26,560 km semi-major axis (20,200 km altitude)
  - The orbital period is approximately 12 hours
- Less than 0.01 eccentricity (near circular)
- 55 degree inclination
- 6 orbital planes with at least 4 satellites in each plane
  - The ascending nodes of the orbital planes are separated by 60 degree



	GPS YUMA ALMANAC FILE
ID	PRN ID of SV
Health	000 = usable
Eccentricity	This shows the amount of the orbit deviation from circular (orbit). It is the distance between the foci divided by the length of the semi-major axis (our orbits are very circular).
Time of applicability	The number of seconds in the orbit when the almanac was generated. Kind of a time tag.
Orbital Inclination	The angle to which the SV orbit meets the equator (GPS is at approximately 55 degrees). Roughly, the SV's orbit will not rise above approximately 55 degrees latitude. The number is part of an equation: # = π/180 = the true inclination.
Rate of Right Ascension	Rate of change in the measurement of the angle of right ascension as defined in the Right Ascension mnemonic.
SQRT(A) Square Root of Semi-Major Axis	This is defined as the measurement from the center of the orbit to either the point of apogee or the point of perigee.
Right Ascension at Time of Almanac (TOA)	Right Ascension is an angular measurement from the vernal equinox ((OMEGA)$_0$).
Argument of Perigee	An angular measurement along the orbital path measured from the ascending node to the point of perigee, measured in the direction of the SV's motion.
Mean Anomaly	Angle (arc) traveled past the longitude of ascending node (value = 0±180 degrees). If the value exceeds 180 degrees, subtract 360 degrees to find the mean anomaly. When the SV has passed perigee and heading towards apogee, the mean anomaly is positive. After the point of apogee, the mean anomaly value will be negative to the point of perigee.
Af0	SV clock bias in seconds.
Af1	SV clock drift in seconds per seconds.
Af2	GPS week (0000–1023), every 7 days since 1999 August 22.
GPS Week	

Example of Yuma Almanac File for GPS

- ******** Week 887 almanac for PRN-01 ********
- ID : 01
- Health : 000
- Eccentricity : 0.5854606628E-002
- Time of Applicability(s) : 589824.0000
- Orbital Inclination(rad) : 0.9652777840
- Rate of Right Ascen(r/s) : -0.7714607059E-008
- SQRT(A) (m 1/2) : 5153.593750
- Right Ascen at Week(rad) : 0.2492756606E+001
- Argument of Perigee(rad) : 0.531310874
- Mean Anom(rad) : 0.3110215331E+001
- Af0(s) : 0.3147125244E-004
- Af1(s/s) : 0.0000000000E+000
- Week : 887

http://qz-vision.jaxa.jp/USE/en/almanac

https://celestrak.com/GPS/almanac/Yuma/definition.asp

Perturbation Forces

- Satellite orbit will be an ellipse only if treating each of satellite and Earth as a point mass.
- In reality, Earth's gravitational field is not a point mass.
- Main force acting on GNSS satellites is Earth's central gravitational force, but there are many other significant perturbations.
  - Non sphericity of the Earth's gravitational potential
  - Third body effect
    - Direct attraction of Moon and Sun
  - Solar radiation pressure
    - Impact on the satellite surfaces of photons emitted by the Sun

Accelerations Acting on GNSS Satellites

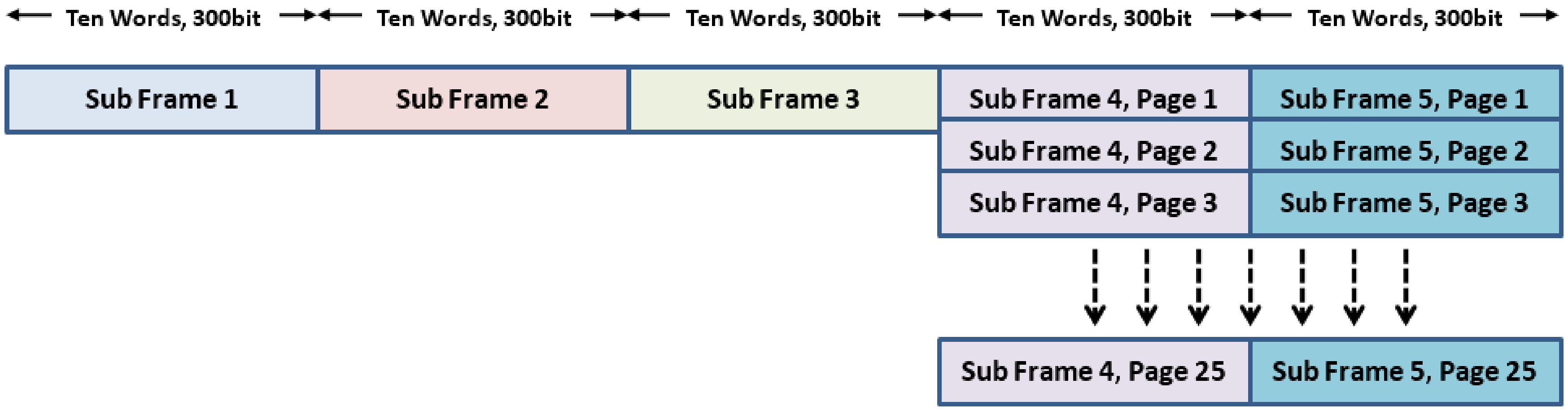
Term	Acceleration [m/s^2]
Earth’s central gravity	0.56
Flatness of the Earth (J2)	5×10^{-5}
Other gravity	3×10^{-7}
Moon and Sun	5×10^{-6}
Solar Radiation Pressure	10^{-7}

Effects of SRP on GNSS satellite position: 5~10 m

Satellite orbit in Navigation Message

- **Broadcast ephemeris**
  - Kepler orbit parameters and satellite clock corrections
  - 9 orbit perturbation corrections parameters
  - 2 m satellite position accuracy for 2 hours
  - Each GNSS satellite broadcasts only its own ephemeris data
- **Almanac**
  - Kepler orbit parameters and satellite clock corrections
  - Less accurate but valid for up to several months
  - Each GNSS satellite broadcasts almanac data for all satellites in the constellation

GPS L1C/A Signal NAV MSG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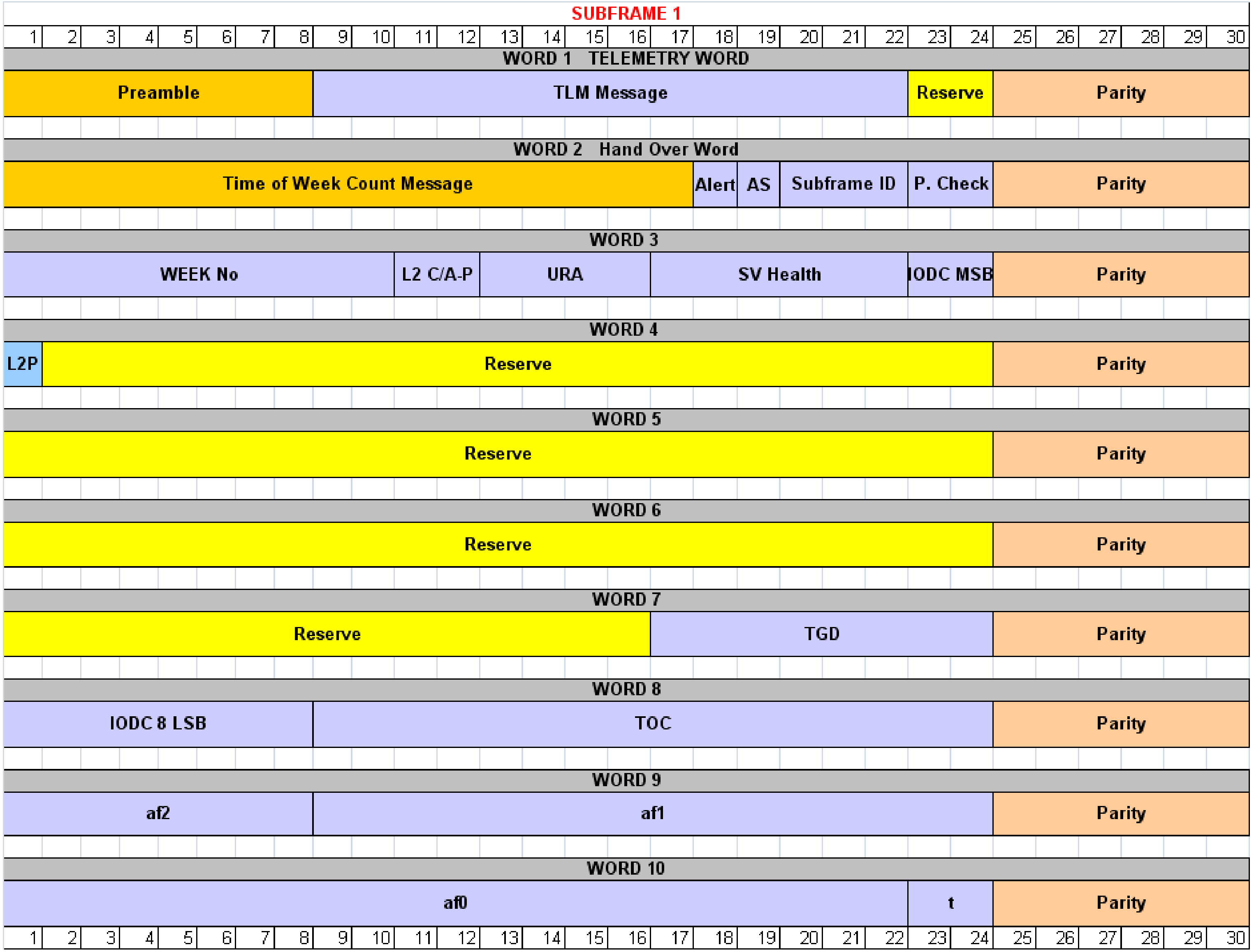
Sub Frame 1
Sub Frame 2
Sub Frame 3
Sub Frame 4, Page 1
Sub Frame 5, Page 1
Sub Frame 4, Page 2
Sub Frame 5, Page 2
Sub Frame 4, Page 3
Sub Frame 5, Page 3
Sub Frame 4, Page 25
Sub Frame 5, Page 25

Navigation Message, Sub-frame 1



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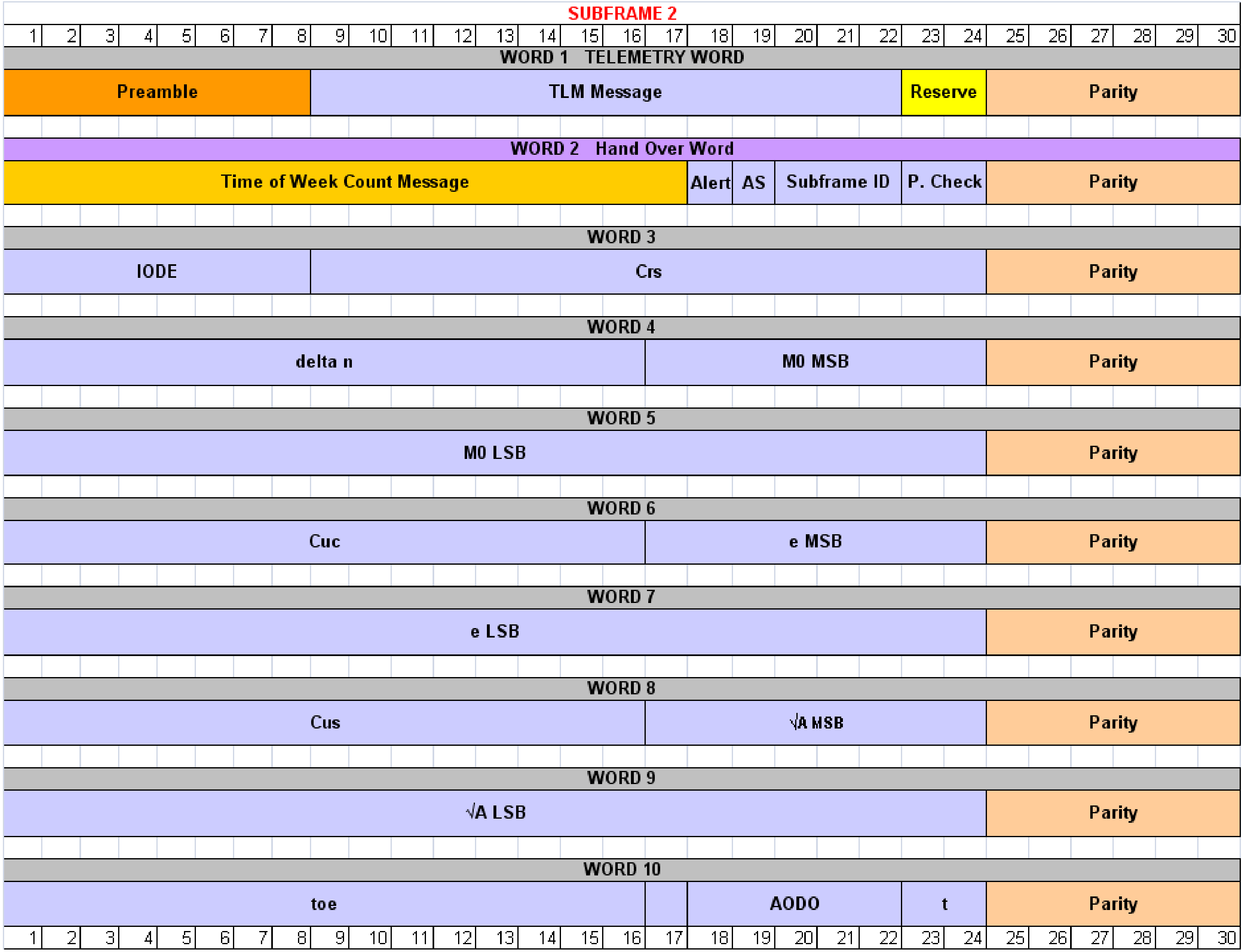
GPS L1C/A Signal NAV MSG, Sub-frame 2



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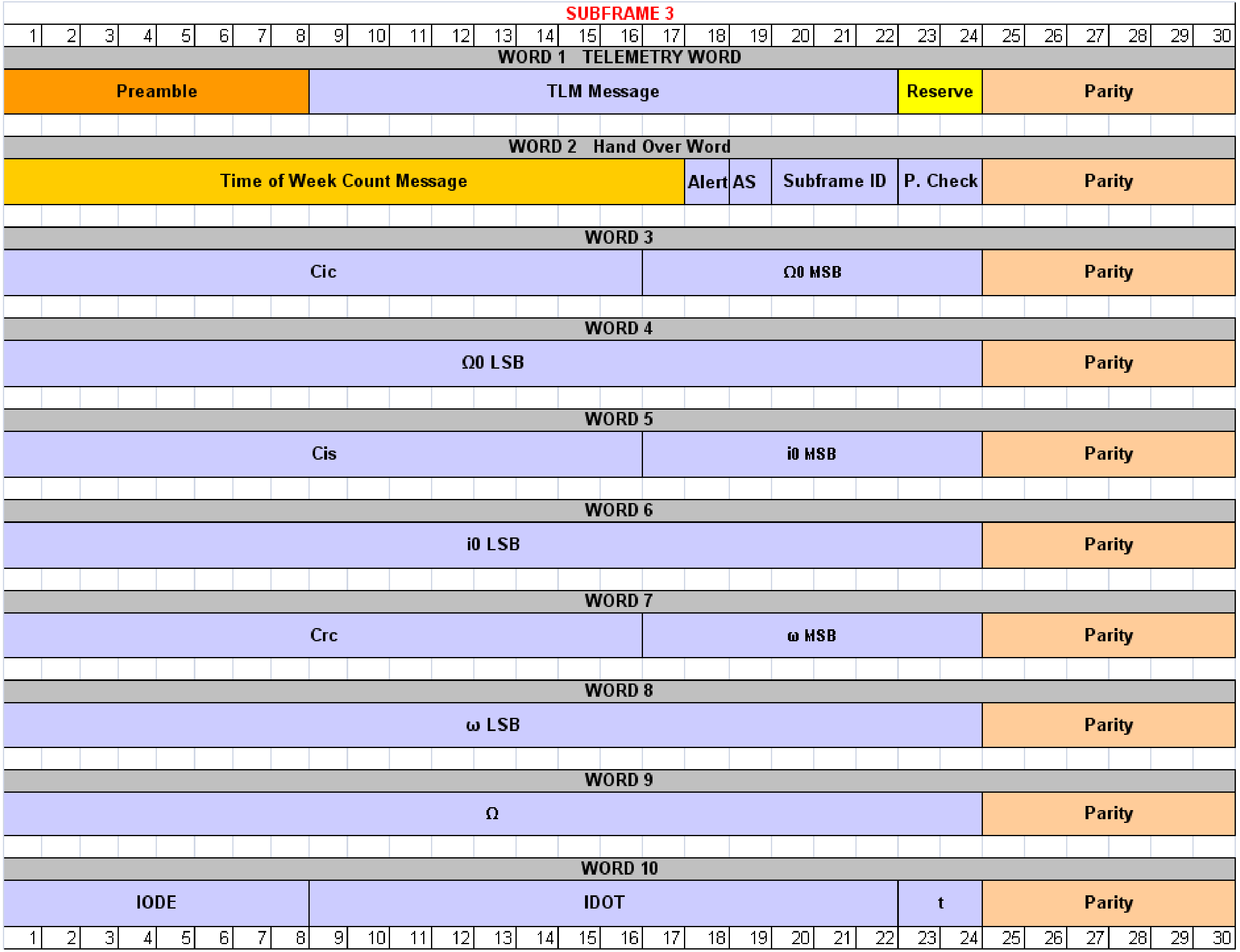

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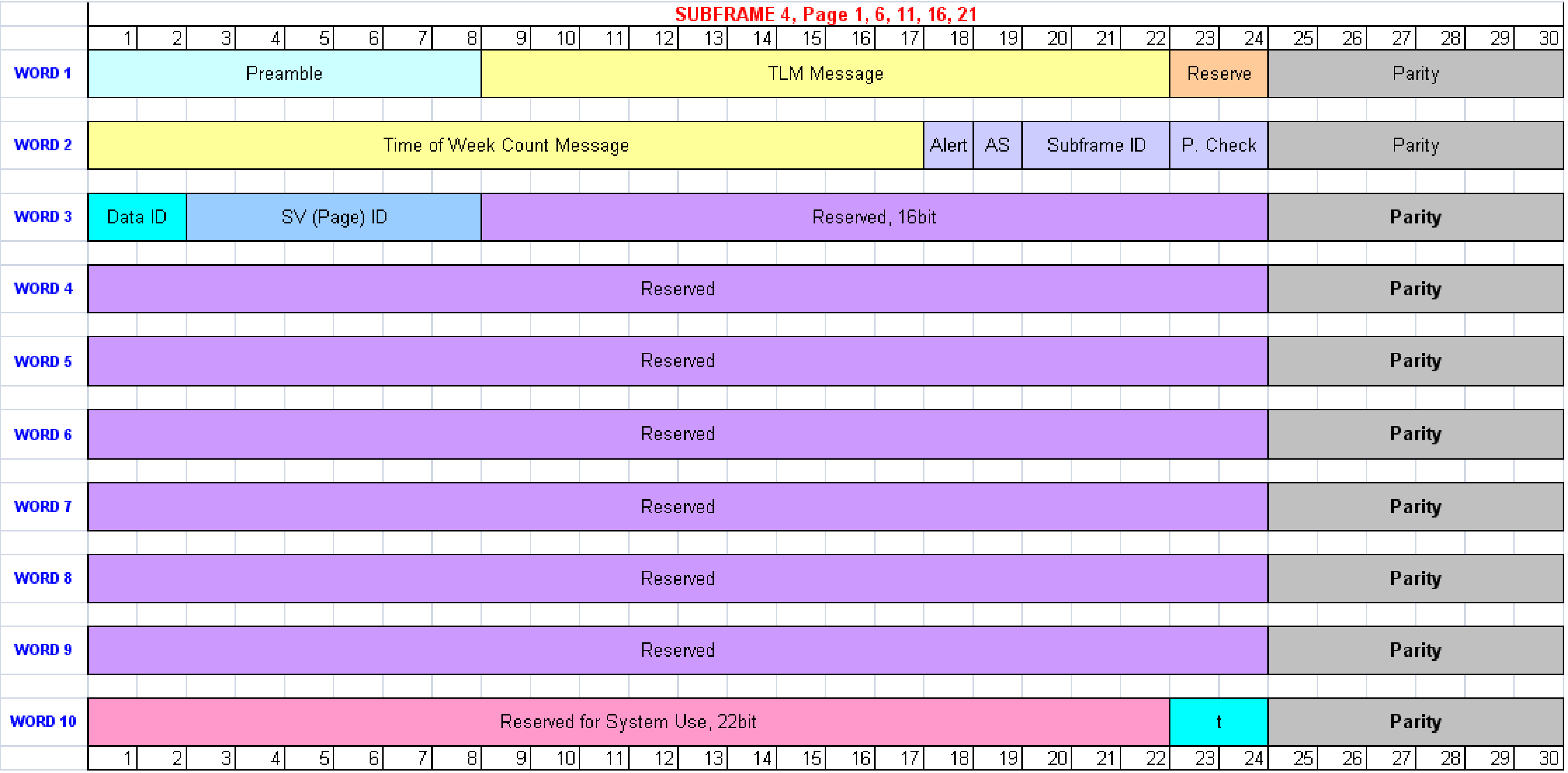
GPS L1C/A Signal NAV MSG, Sub-frame 4 Page 1,6,11,16,21



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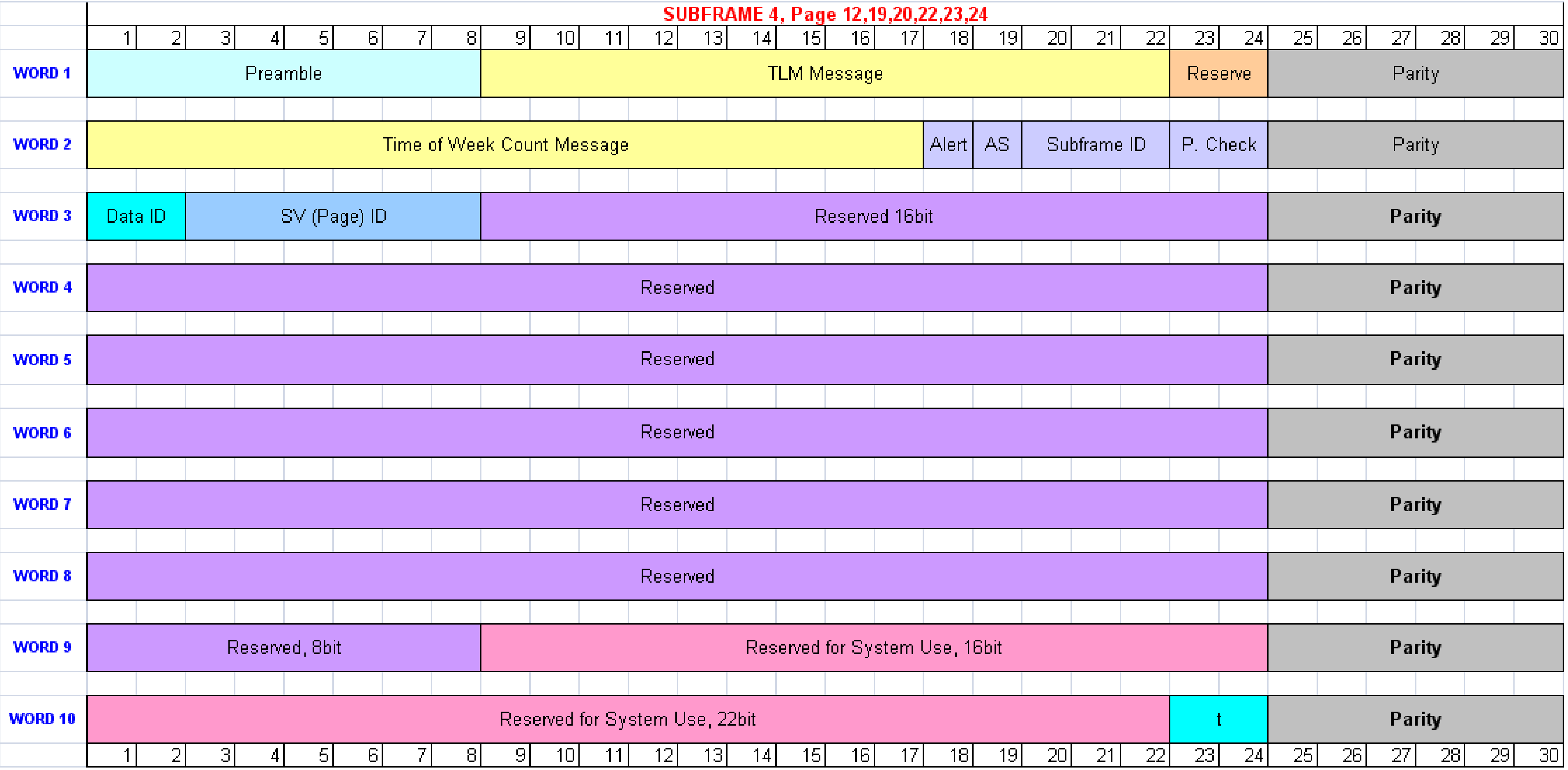
GPS L1C/A Signal NAV MSG, Sub-frame 4 Page 12,19,20,22,23,24



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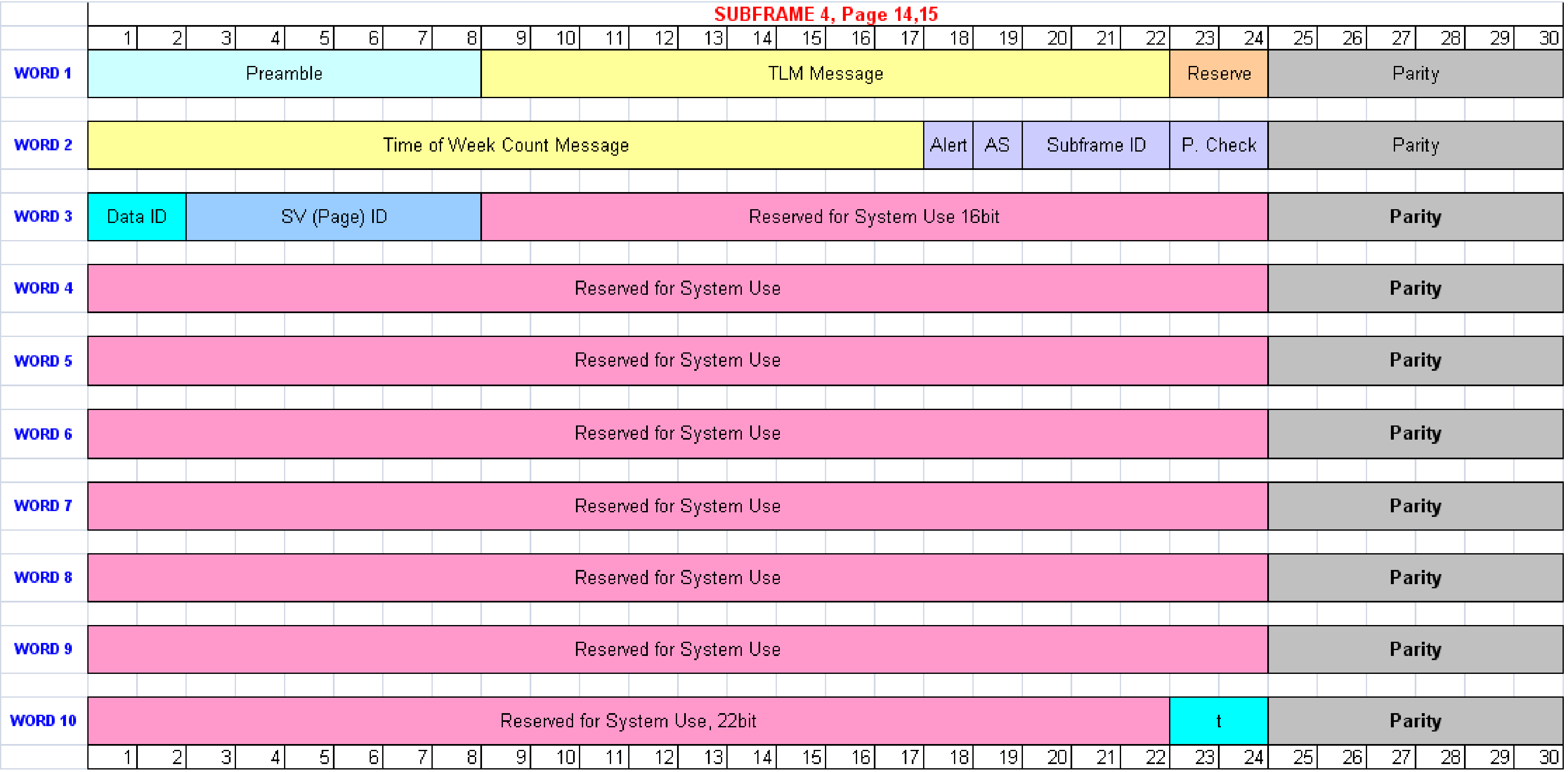
GPS L1C/A Signal NAV MSG, Sub-frame 4, Page 14, 15



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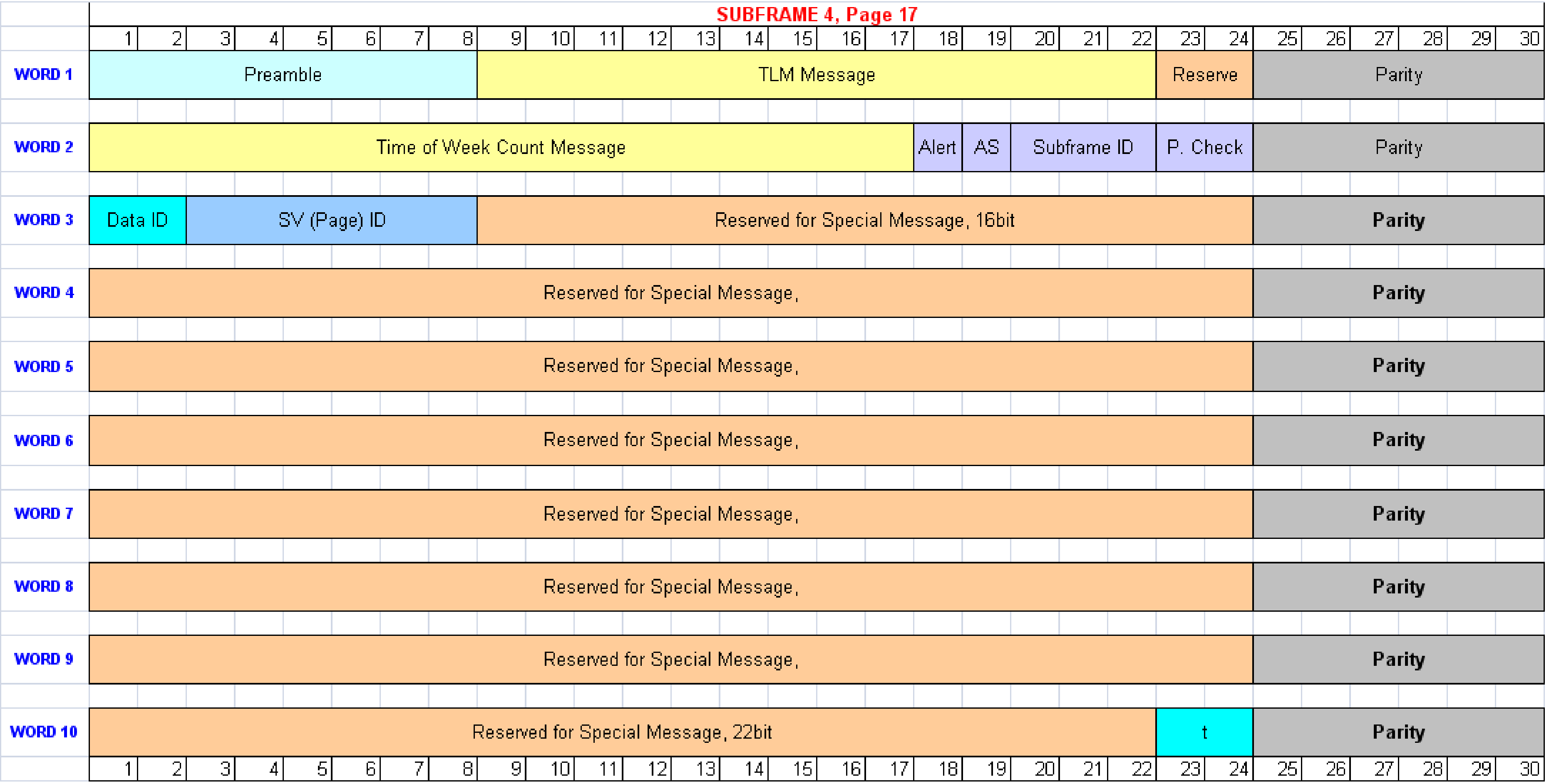
GPS L1C/A Signal NAV MSG, Sub-frame 4, Page 17



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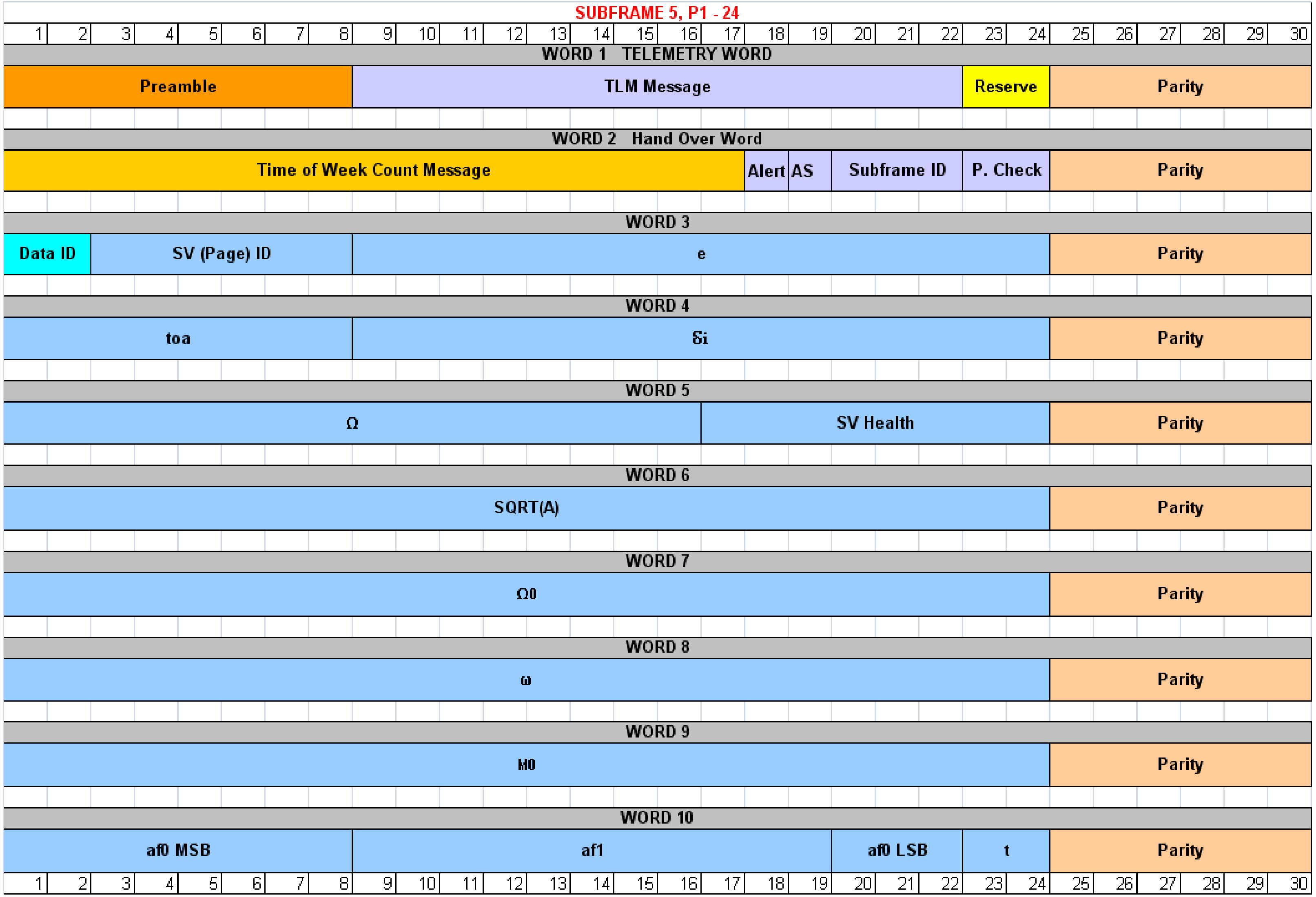
GPS L1C/A Signal NAV MSG, Sub-frame 5



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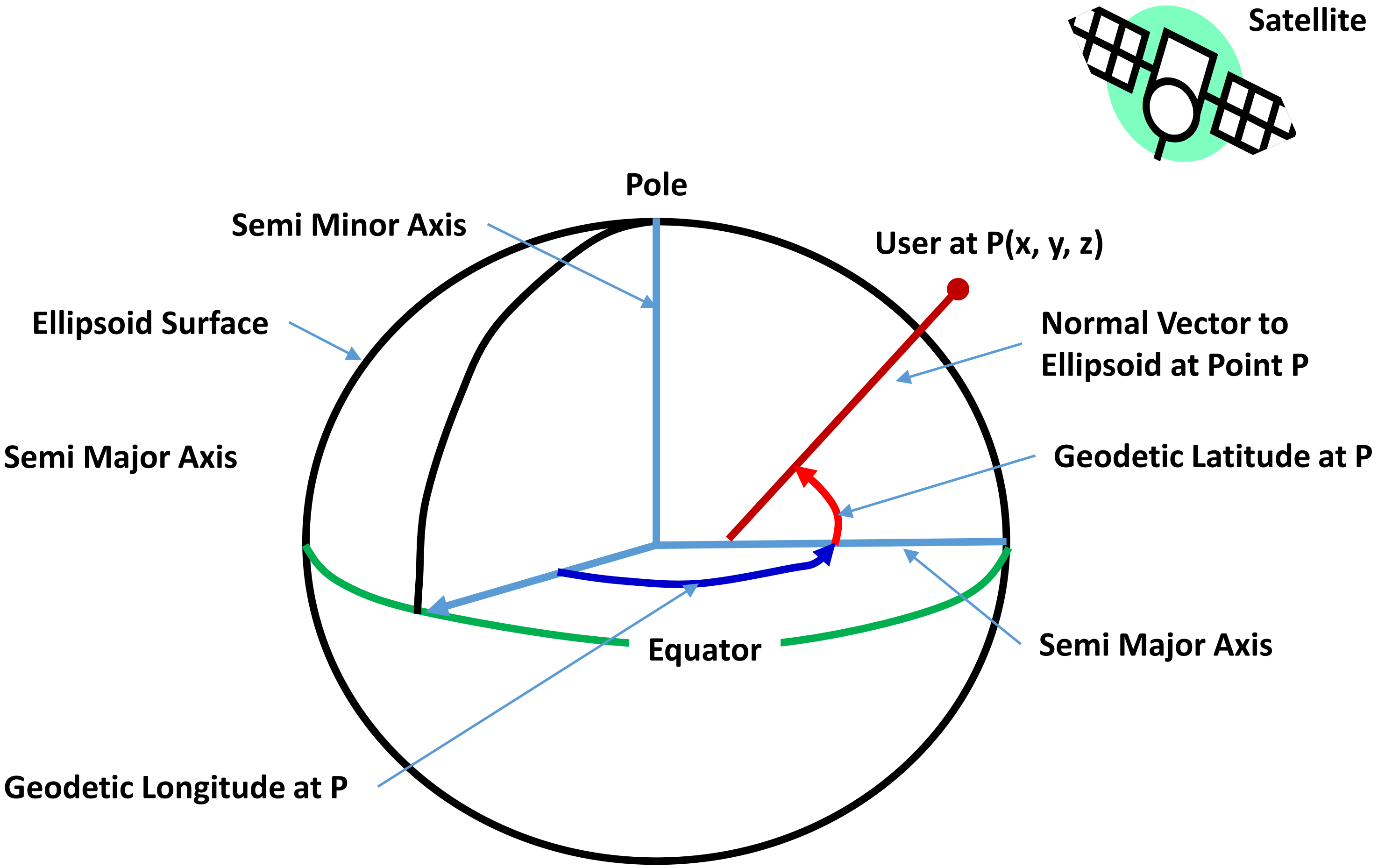
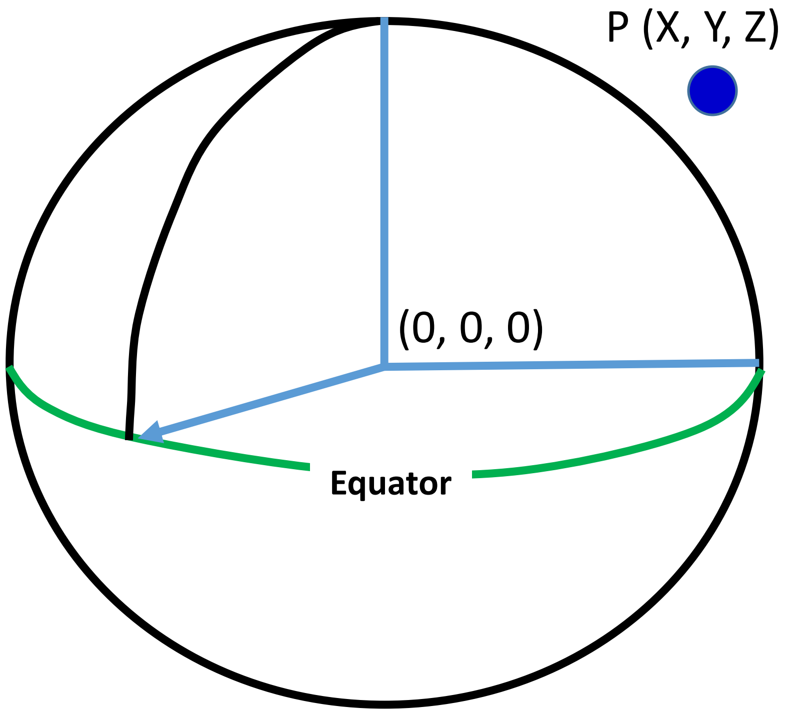
Coordinate System

Geodetic Coordinate System

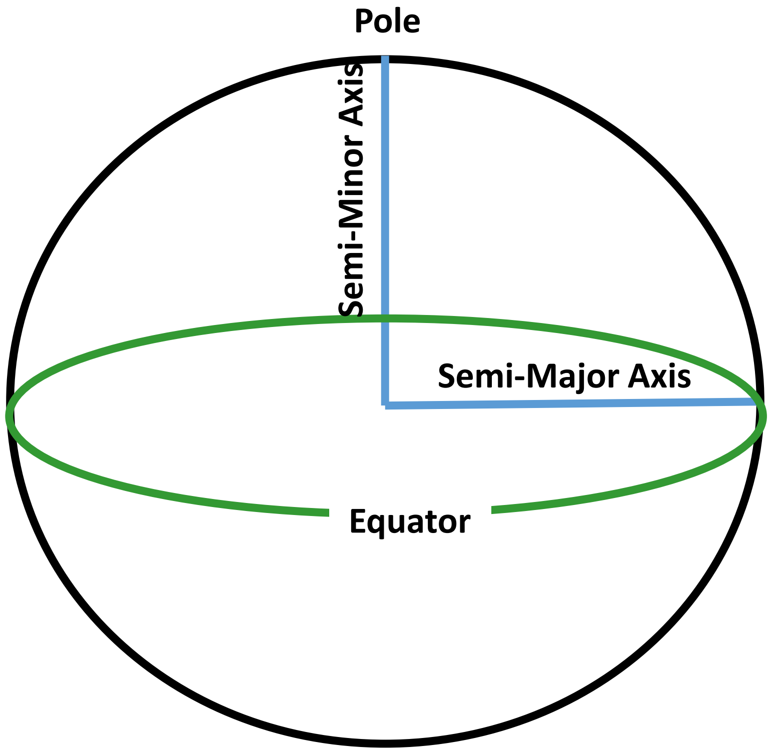


ECEF (Earth Centered, Earth Fixed)

ECEF Coordinate System is expressed by assuming the center of the earth coordinate as (0, 0, 0)



Geodetic Datum: Geometric Earth Model



WGS-84 Geodetic Datum Ellipsoidal Parameters

Semi-Minor Axis, b = 6356752.3142m

Semi-Major Axis, a = 6378137.0m

Flattening, f = (a-b)/a
= 1/298.257223563

First Eccentricity Square = e^2 = 2f-f^2
= 0.00669437999013

Coordinate Conversion from ECEF to Geodetic and vice versa

Geodetic Latitude, Longitude & Height to ECEF (X, Y, Z)

$$X = (N+h)\cos\varphi\cos\lambda$$

$$Y = (N+h)\cos\varphi\sin\lambda$$

$$Z = [N(1-e^2)+h]\sin\varphi$$

$\varphi = Latitude$
$\lambda = Longitude$
H = Height above Ellipsoid

ECEF (X, Y, Z) to Geodetic Latitude, Longitude & Height

$$\varphi = \text{atan}\left(\frac{Z+e^2 b\, sin^3\theta}{p-e^2 a cos^3\theta}\right)$$

$$\lambda = \text{atan2}(Y, X)$$

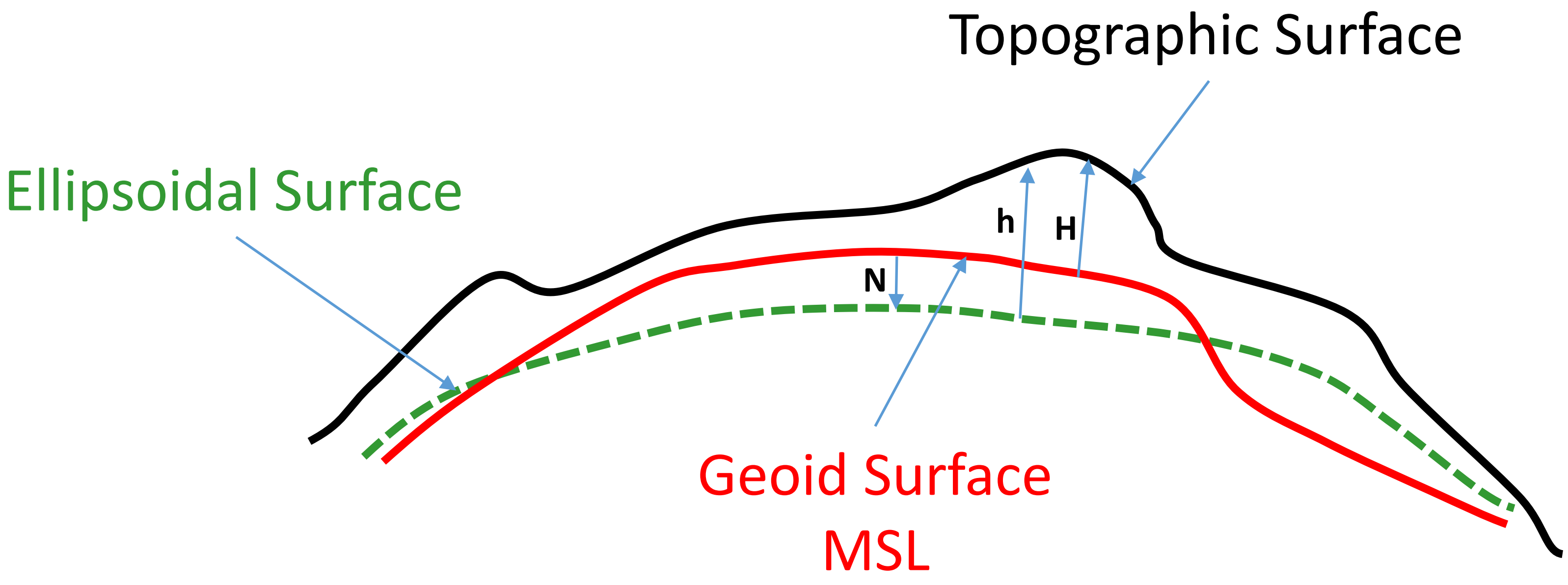
$$h = \frac{P}{\cos\varphi} - \mathrm{N}(\varphi)$$

$$P = \sqrt{x^2+y^2}$$

$$\theta = atan\left(\frac{Za}{Pb}\right)$$

$$N(\varphi) = \frac{a}{\sqrt{1-e^2 sin^2\varphi}}$$

Topographic, Ellipsoidal & Geoid Height



Topographic Height (H) = Ellipsoidal Height (h) - Geoid Height (N)

Position Output

Pseudorange equation

Perfect World: $r = c\left(t_R - t^S\right)$

Real World:

Tropospheric delay

Receiver clock error

Multipath

$$\rho = r + c\left(\delta t_R - \delta t^S\right) + I + T + M + \xi$$

Satellite clock error

Ionospheric delay

Thermal noise

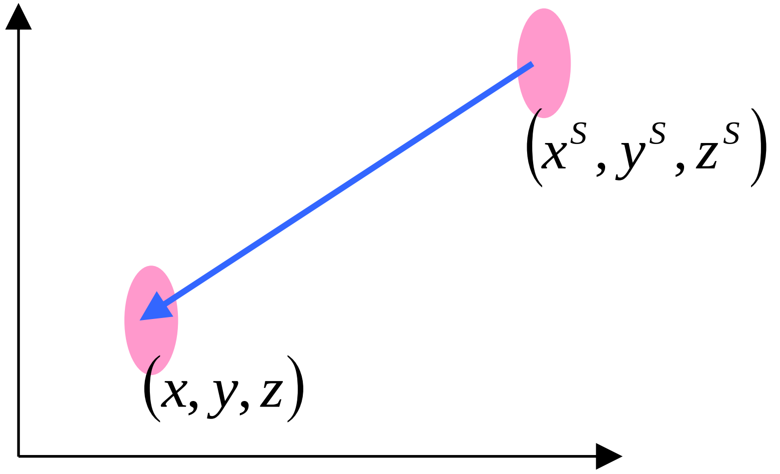
Simplify… $\rho = r + c\left(\delta t_R - \delta t^S\right) + \varepsilon$

Range Equation

Satellite position at signal transmission time: $\left(x^S, y^S, z^S\right)$

Receiver position at signal reception time: $\left(x, y, z\right)$

$$r = \sqrt{\left(x - x^S\right)^2 + \left(y - y^S\right)^2 + \left(z - z^S\right)^2}$$



Pseudorange model

$$\rho = \underbrace{\sqrt{(x - x^S)^2 + (y - y^S)^2 + (z - z^S)^2}}_{r} + c(\delta t_R - \delta t^S) + \varepsilon$$

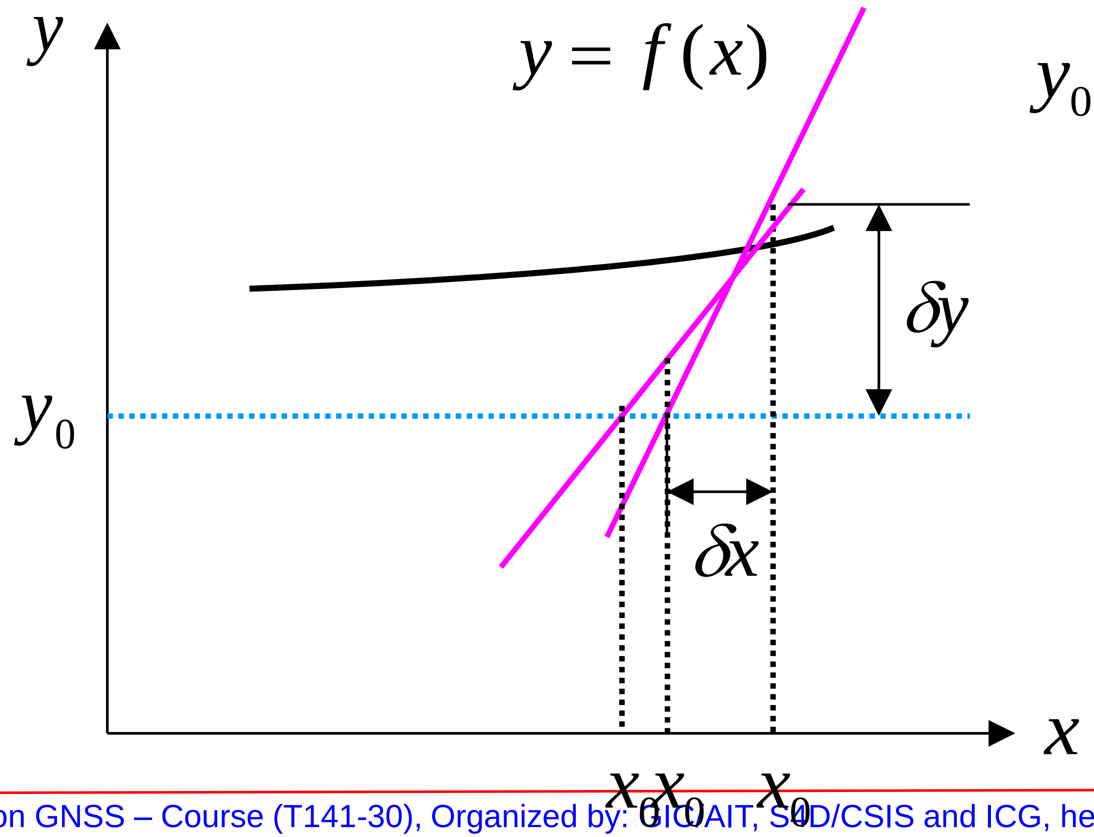
- ○ Given satellite position & clock in navigation message
- ○ Unknown receiver position & clock
- ○ Estimate optimal solution to minimize the error

Nonlinear Optimization Problem

- We have *n* simultaneous nonlinear equations from *n* pseudorange observations.
- We need at least 4 independent observations in order to determine 4 unknown parameters.
- In general, even a single nonlinear equation cannot be solved without some iterative method by generating a sequence of approximate solutions.

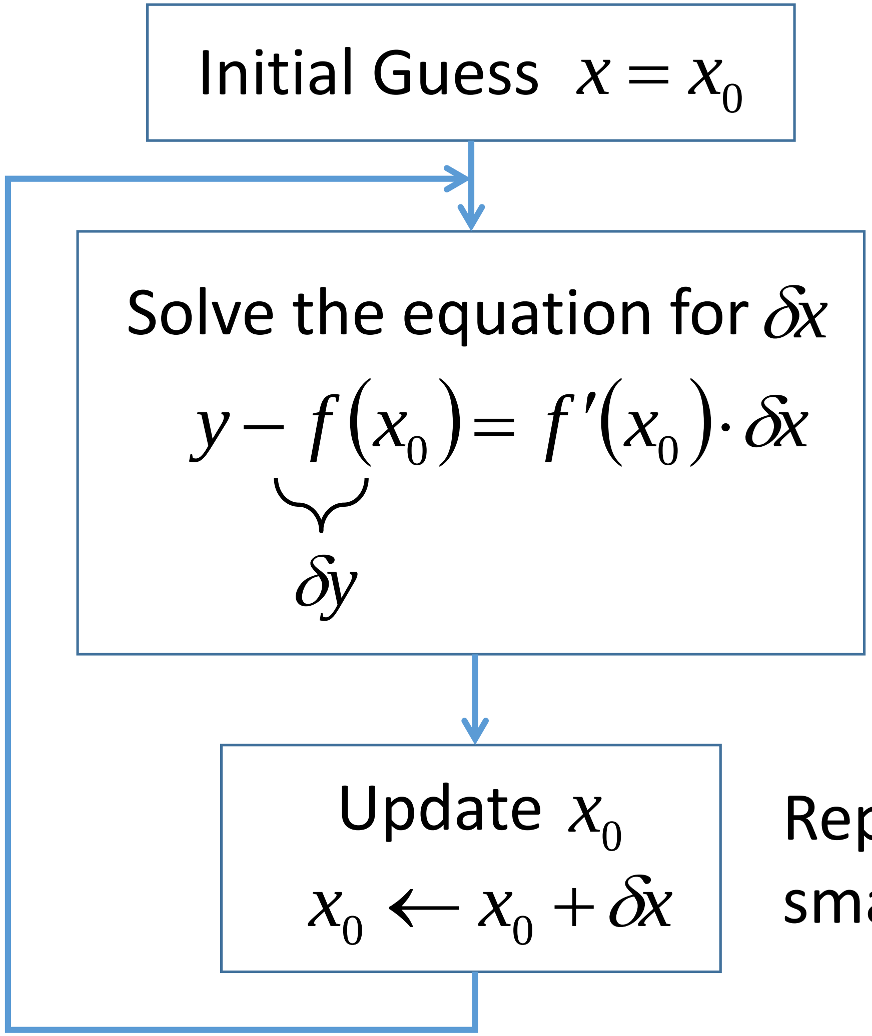
Newton-Raphson Method

Find successively better approximation $x = x_0$ satisfying $y = y_0$ of a nonlinear equation $y = f(x)$



$$\underbrace{y_0 - f(x_0)}_{\delta y} = \underline{f'(x_0)} \cdot \delta x$$

Newton-Raphson Algorithm



Initial Guess $x = x_0$

Solve the equation for δx

$$\underbrace{y - f(x_0)}_{\delta y} = f'(x_0)\cdot\delta x$$

δy: Residual

Update x_0

$$x_0 \leftarrow x_0 + \delta x$$

Repeat until a sufficiently small value δx is reached.

Pseudorange Equation

$$\rho = \underbrace{\sqrt{\left(x - x^S\right)^2 + \left(y - y^S\right)^2 + \left(z - z^S\right)^2}}_{r} + \underbrace{c\delta t}_{b} - c\delta t^S$$

$$= f\left(x, y, z, b\right)$$

For given observation $\rho = \rho_0$
Linearize around the initial solution $\left(x_0, y_0, z_0, b_0\right)$
Obtain the update $\left(\delta x, \delta y, \delta z, \delta b\right)$

Linearization

Partial derivatives with respect to each unknown parameter:

$$\frac{\partial f}{\partial x} = \frac{x - x^S}{r}, \quad \frac{\partial f}{\partial y} = \frac{y - y^S}{r}, \quad \frac{\partial f}{\partial z} = \frac{z - z^S}{r}, \quad \frac{\partial f}{\partial b} = 1$$

Linearized pseudorange residual equation:

$$\underbrace{\rho_0 - f(x_0, y_0, z_0, b_0)}_{\delta\rho} = \frac{x_0 - x^S}{r_0}\bigcirc\!\!\!\!\!\delta x + \frac{y_0 - y^S}{r_0}\bigcirc\!\!\!\!\!\delta y + \frac{z_0 - z^S}{r_0}\bigcirc\!\!\!\!\!\delta z + \bigcirc\!\!\!\!\!\delta b$$

Vector Description

$$\delta\rho = \underbrace{\begin{bmatrix} \dfrac{x_0 - x^S}{r_0} & \dfrac{y_0 - y^S}{r_0} & \dfrac{z_0 - z^S}{r_0} & 1 \end{bmatrix}}_{\mathbf{h}} \underbrace{\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta b \end{bmatrix}}_{\mathbf{x}} + \varepsilon$$

We need at least 4 linearly independent equations in order to determine 4 unknown parameters.

Simultaneous equations

$$\underbrace{\begin{bmatrix}\delta\rho^1\\ \delta\rho^2\\ \vdots\\ \delta\rho^N\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix}\mathbf{h}^1\\ \mathbf{h}^2\\ \vdots\\ \mathbf{h}^N\end{bmatrix}}_{\mathbf{H}}\mathbf{x} + \underbrace{\begin{bmatrix}\varepsilon^1\\ \varepsilon^2\\ \vdots\\ \varepsilon^N\end{bmatrix}}_{\mathbf{e}} \qquad N \geq 4$$

$$(N\times 1) \qquad (N\times 4)\,(4\times 1) \qquad (N\times 1)$$

Residual vector

$$\boxed{\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}}$$

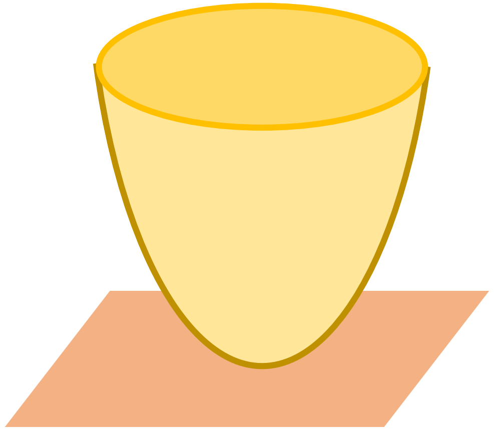
Observation matrix   State vector

Least squares problem

For given $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$, find $\mathbf{x} = \hat{\mathbf{x}}$ to minimize $\mathbf{e}^T\mathbf{e}$.

Performance Index:

$$
\begin{aligned}
J &= \mathbf{e}^T\mathbf{e} \\
&= (\mathbf{y} - \mathbf{H}\mathbf{x})^T(\mathbf{y} - \mathbf{H}\mathbf{x}) \\
&= \mathbf{y}^T\mathbf{y} - \mathbf{y}^T\mathbf{H}\mathbf{x} - \mathbf{x}^T\mathbf{H}^T\mathbf{y} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x} \\
&= \mathbf{y}^T\mathbf{y} - 2\mathbf{x}^T\mathbf{H}^T\mathbf{y} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x}
\end{aligned}
$$



Find $\mathbf{x} = \hat{\mathbf{x}}$ to minimize $J \Leftrightarrow \left.\dfrac{\partial J}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}} = 0$

Least squares solution

Partial derivatives of a scalar function w.r.t the state vector:

(1) $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} = \mathbf{x}^T\mathbf{a}$ then $\dfrac{\partial f}{\partial \mathbf{x}} = \mathbf{a}$

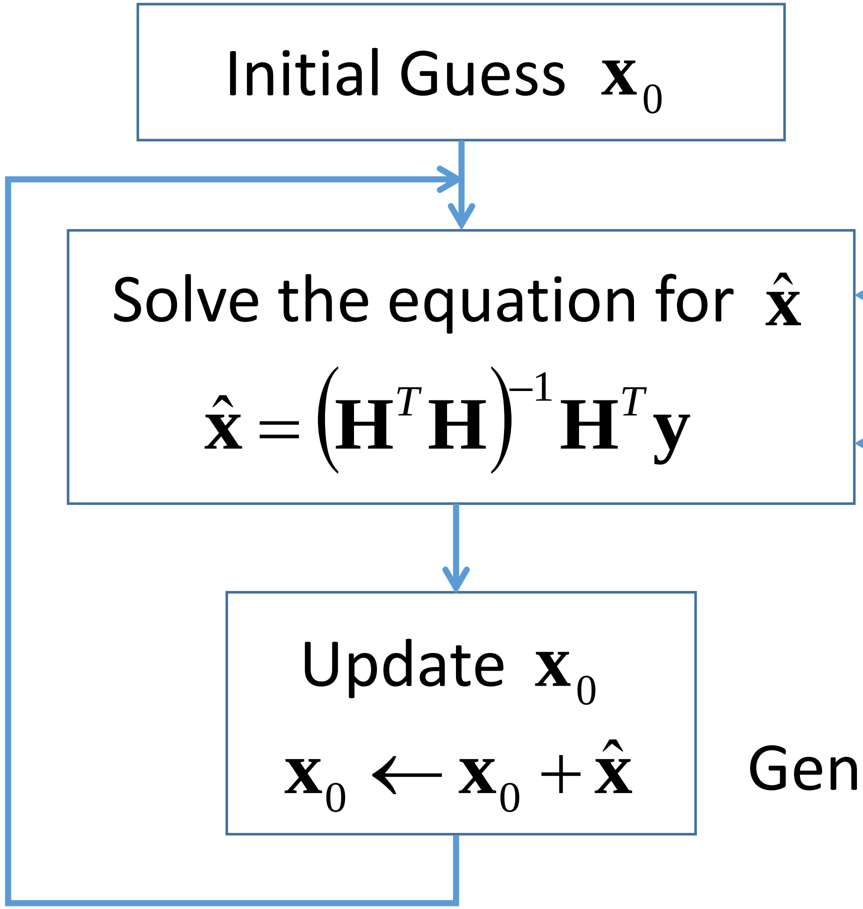
(2) $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ then $\dfrac{\partial f}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$

for all symmetric matrix $\mathbf{A}$

Find $\mathbf{x} = \hat{\mathbf{x}}$ to satisfy $\dfrac{\partial J}{\partial \mathbf{x}} = -2\mathbf{H}^T\mathbf{y} + 2\mathbf{H}^T\mathbf{H}\mathbf{x} = \mathbf{0}$

➡ $\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{y}$

GNSS Positioning Calculation



Initial Guess $\mathbf{x}_0$

Solve the equation for $\hat{\mathbf{x}}$

$$\hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{y}$$

Update $\mathbf{x}_0$

$$\mathbf{x}_0 \leftarrow \mathbf{x}_0 + \hat{\mathbf{x}}$$

Even the north pole is a good initial guess!

Pseudorange ρ

Satellite position $\left(x^S, y^S, z^S\right)$

Satellite clock δt^S

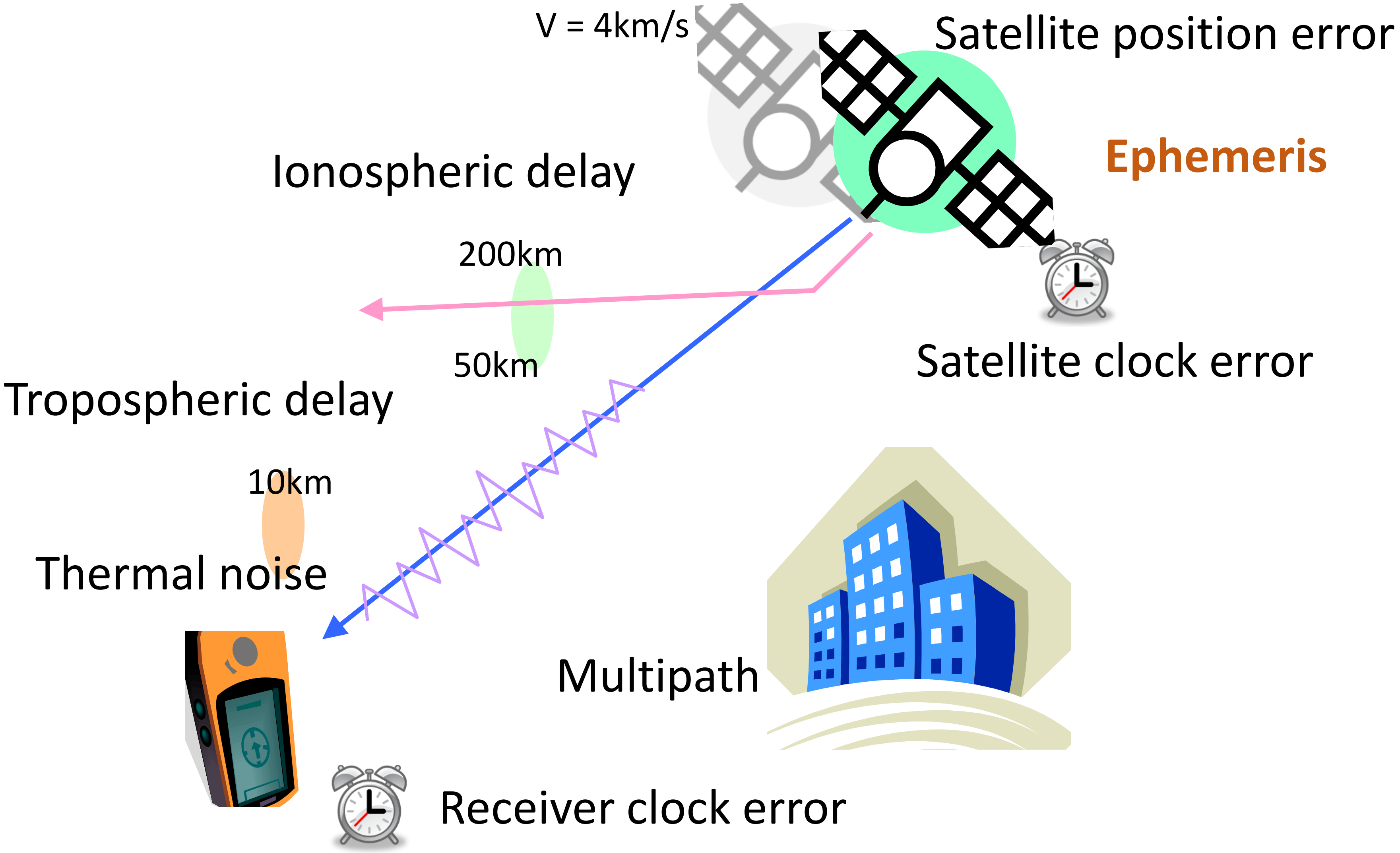
Generally takes a few iterations.

Error Budget

- The positioning accuracy depends on the magnitude of error in the individual pseudorange measurement.

Source	Error	DGPS
Satellite orbit error	1 ~2m	0
Satellite clock error	1 m	0
Ionospheric delay	4~10 m	Can be minimized to <1m
Tropospheric delay	1~2 m	
Thermal noise	1 m	Can't be removed
Multipath	1m or more	

Error sources

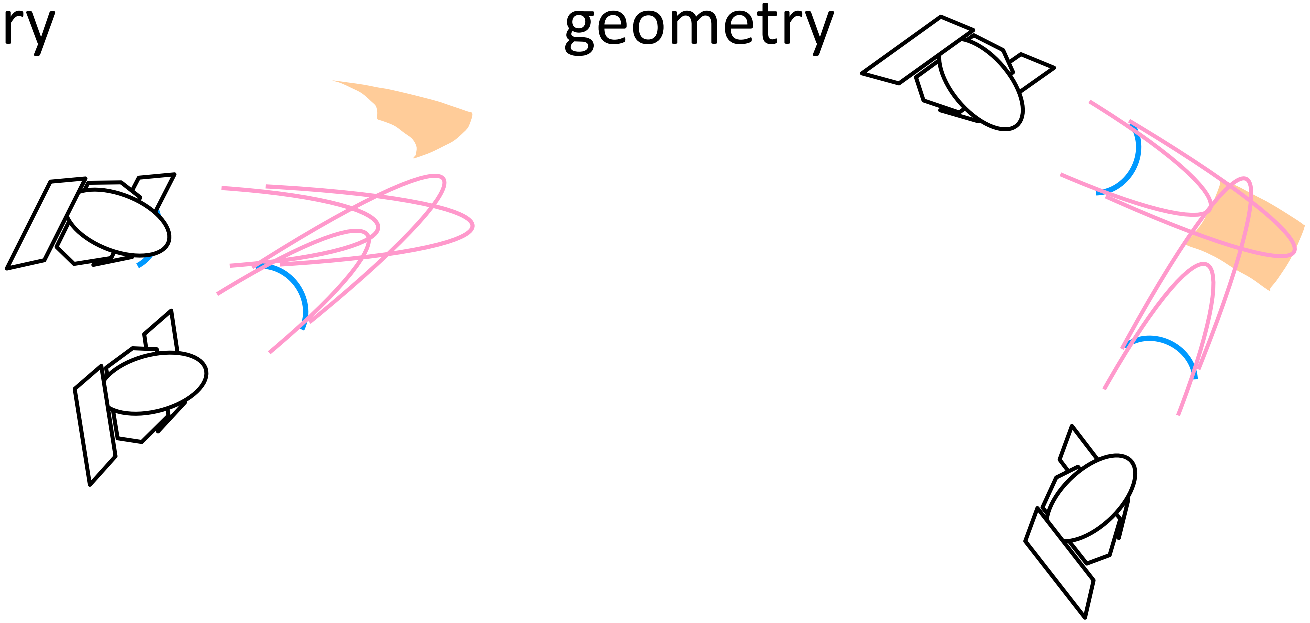


Satellite geometry and positioning error

- The positioning accuracy also depends on the geometric configuration of the satellites.

Bad geometry

Good geometry



Dilution of precision (DOP)

$$\mathbf{G} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1} = \begin{bmatrix} g_x & \cdot & \cdot & \cdot \\ \cdot & g_y & \cdot & \cdot \\ \cdot & \cdot & g_z & \cdot \\ \cdot & \cdot & \cdot & g_b \end{bmatrix}$$

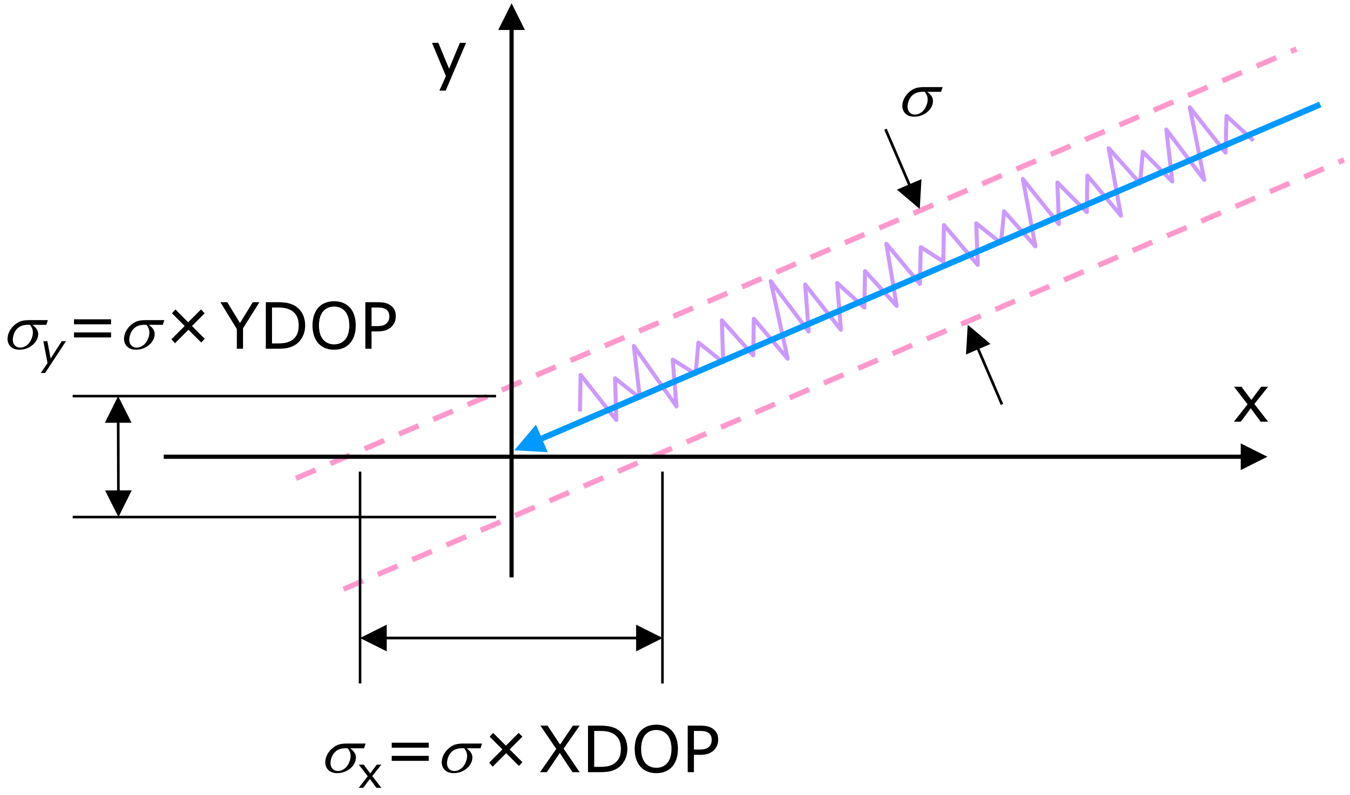
Position DOP: $\mathrm{PDOP} = \sqrt{g_x + g_y + g_z}$

Time DOP: $\mathrm{TDOP} = \sqrt{g_b}$

Geometric DOP: $\mathrm{GDOP} = \sqrt{g_x + g_y + g_z + g_b}$

DOP and positioning accuracy

Accuracy of any measurement is proportionately dependent on the DOP value. This means that if DOP value doubles, the resulting position error increases by a factor of two.



Data Formats: NMEA, RINEX

NMEA Data Format

- NMEA
  - National Marine Electronics Association format to share position, velocity, satellite visibility and many other formats
  - ASCII file with pre-defined headers
    - For example “$GP” for GPS Related Data
    - $GPGSV for GPS Satellite Visibility
    - “$GN” is used for GNSS

NMEA Data Format

NMEA: National Marine Electronics Association

GGA - Fix data which provide 3D location and accuracy data.

$GPGGA,123519,4807.038,N,01131.000,E,1,08,0.9,545.4,M,46.9,M,,*47

Where: GGA Global Positioning System Fix Data
123519 Fix taken at 12:35:19 UTC
4807.038,N Latitude 48 deg 07.038' N
01131.000,E Longitude 11 deg 31.000' E
1 Fix quality:
0 = invalid ,
1 = GPS fix (SPS),
2 = DGPS fix,
3 = PPS fix,
4 = Real Time Kinematic
5 = Float RTK
6 = estimated (dead reckoning) (2.3 feature)
7 = Manual input mode
8 = Simulation mode
08 Number of satellites being tracked
0.9 Horizontal dilution of position
545.4,M Altitude, Meters, above mean sea level
46.9,M Height of geoid (mean sea level) above WGS84 ellipsoid
(empty field) time in seconds since last DGPS update (empty field) DGPS station ID number *47 the checksum data, always begins with *

RINEX Data Format

- Receiver Independent Exchange Format
- Basically Two File Types
  - “*.*N” file for Satellite and Ephemeris Related data
  - “*.*O” file for Signal Observation Data like Pseudorange, Carrier Phase, Doppler, SNR etc

Example of RINEX *.*N file

- 2.12 N RINEX VERSION / TYPERDE MCS 20160826 042004 UTC
  PGM / RUN BY / DATEGPSA 5.5879D-09 4.7432D-09 -6.0392D-09 -3.8447D-09 IONOSPHERIC CORRGPSB
  7.7824D+04 1.0430D+04 -6.6402D+03 -8.4545D+03 IONOSPHERIC CORRGPUT -9.3132257462D-10-
  1.776356839D-15 61440 1912 TIME SYSTEM CORR 17 LEAP SECONDS
  END OF HEADER14 16 8 25 23 59 44.0-3.174087032676D-05-1.932676241267D-12 0.000000000000D+00
  5.000000000000D+01 3.028125000000D+01 4.619478134092D-09 6.415934976073D-01
  1.654028892517D-06 8.938927436247D-03 9.655952453613D-06 5.154050765991D+03
  4.319840000000D+05-4.656612873077D-08-1.624672846442D+00 5.960464477539D-08
  9.631013196872D-01 1.981562500000D+02-1.956144411918D+00-8.141410551000D-09 -
  4.296607542379D-10 1.000000000000D+00 1.911000000000D+03 0.000000000000D+00
  2.000000000000D+00 0.000000000000D+00-9.313225746155D-09 5.000000000000D+01
  4.248300000000D+05 4.000000000000D+00

Example of RINEX *.*q file

- 2.12 N J RINEX VERSION / TYPERDE MCS 20160826 042004 UTC PGM / RUN BY / DATEQZSA 3.3528D-08 -1.6364D-07 3.0800D-07 -2.3068D-07 IONOSPHERIC CORRQZSB 1.0854D+05 8.3443D+04 -8.4994D+05 2.6843D+05 IONOSPHERIC CORRQZUT 2.1420419216D-08 1.776356839D-15 90112 1912 TIME SYSTEM CORR 17 LEAP SECONDS END OF HEADERJ 1 16 8 26 0 0 0.0 1.034699380398D-05 3.160494088661D-11 0.000000000000D+00 9.500000000000D+01-6.181250000000D+01 3.210848030423D-09 6.710743529404D-01 -2.145767211914D-06 7.513544522226D-02 4.915520548820D-06 6.493574121475D+03 4.320000000000D+05-9.723007678986D-07-2.898287313277D+00-8.530914783478D-07 7.114301638419D-01 2.543750000000D+01-1.564723996692D+00-3.286208312338D-09 3.560862609935D-10 2.000000000000D+00 1.911000000000D+03 0.000000000000D+00 2.000000000000D+00 1.000000000000D+00-5.587935447693D-09 9.500000000000D+01 4.284300000000D+05 2.000000000000D+00

Example of RINEX *.*O File

- 3.03 OBSERVATION DATA M (MIXED) RINEX VERSION / TYPENetR9 5.10 Receiver Operator 03-AUG-16 00:00:00
  PGM / RUN BY / DATECREF0001 MARKER NAMEGEODETIC MARKER TYPE
  OBS AGENCY OBSERVER / AGENCY 5536R50102 Trimble NetR9 5.10 REC # / TYPE /
  VERS TRM57971.00 NONE ANT # / TYPE 0.0000 0.0000 0.0000 APPROX POSITION XYZ
  0.0001 0.0000 0.0000 ANTENNA: DELTA H/E/NG 9 C1C L1C S1C C2W L2W S2W C2X L2X S2X SYS / # /
  OBS TYPESR 9 C1C L1C S1C C1P L1P S1P C2C L2C S2C SYS / # / OBS TYPESE 9 C1X L1X S1X C7X L7X S7X C8X L8X S8X
  SYS / # / OBS TYPES 1.000 INTERVAL 2016 8 3 0 0 0.0000000 GPS TIME OF FIRST
  OBSL2C CARRIER PHASE MEASUREMENTS: PHASE SHIFTS REMOVED COMMENT L2C PHASE MATCHES L2 P PHASE
  COMMENT GLONASS C/A & P PHASE MATCH: PHASE SHIFTS REMOVED COMMENT GIOVE-A if present is
  mapped to satellite ID 51 COMMENT GIOVE-B if present is mapped to satellite ID 52 COMMENT DBHZ
  SIGNAL STRENGTH UNIT END OF HEADER > 2016 8 3 0 0 0.0000000 0 15
  0.000000000000G23 22910997.969 6 120398118.969 6 38.700 22911003.211 3 93816706.987 3 18.300G27
  20498538.711 7 107720576.826 7 45.000 20498546.852 4 83938142.552 4 29.500 20498547.680 7 83938139.557 7
  43.900G21 23417862.563 6 123061757.142 6 38.600 23417868.961 2 95892273.957 2 16.300G31 22332200.461 6
  117356474.102 6 40.100 22332207.371 3 91446624.132 3 21.100 22332207.273 6 91446635.131 6 38.200R17
  19246335.906 6 102991051.214 6 40.900 19246335.555 6 102990857.206 6 39.500 19246341.723 6 80104178.556 6
  38.500E22 26811271.836 6 140894162.607 6 37.900 26811279.609 6 107957884.921 6 37.200 26811281.586 6
  106585639.514 6 36.800E30 26058296.672 6 136937242.154 6 39.900 26058305.926 6 104925951.595 6 38.500
  26058308.176 6 103592172.441 6 38.300R14 19701830.117 5 105021857.953 5 34.900 19701829.344 5 105021820.906
  5 33.600 19701838.480 5 81683675.988 5 34.800R18 21955475.016 6 117199783.554 6 36.500 21955474.645 5
  117199814.536 5 35.000 21955482.137 6 91155439.683 6 36.300G08 22841508.133 6 120032929.389 6 39.400
  22841517.746 3 93532209.520 3 18.300 22841518.262 6 93532206.544 6 39.500R24 20876981.063 5 111638735.431
  5 34.400 20876981.367 5 111638615.466 5 32.300 20876986.434 4 86830023.728 4 25.300G09 23668814.758 5
  124380456.384 5 33.600 23668823.629 2 96919879.188 2 14.800 23668824.441 5 96919880.199 5 34.800G26
  21060575.414 7 110674056.882 7 43.300 21060584.641 4 86239571.298 4 26.400 21060584.910 7 86239554.302 7
  43.000G16 20714189.211 7 108853737.965 7 43.100 20714194.789 4 84821163.788 4 25.800R15 19871103.195 6
  106185038.553 6 38.800 19871103.809 6 106185152.572 6 37.100 19871111.785 6 82588483.776 6 37.900> 2016
  8 3 0 0 1.0000000 0 15 0.000000000000